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ON FINITE AMPLITUDE WAVES AT THE BOUNDARY OF SEPARATION OF TWO FLOWS OF A HEAVY IDEAL INCOMPRESSIBLE FLUID*

A.G. PETROV and A.N. CHUVASHOV

A planar Cauchy-Poisson problem of the waves at the interface of two flows of a heavy ideal incompressible fluid which move at different velocities when there is a small homogeneity in the density is investigated. A method is proposed which enables one to investigate not only travelling and standing waves but also the general Cauchy-Poisson problem taking account of the difference in the velocities of the flows. The basis of this method is a linear integral relationship which associates the value of a harmonic function on the boundary of its derivative. A system of integrodifferential equations is obtained for the function which determines the profile of the wave and the discontinuity in the potential at the interface of the flows. The equations of the internal waves which have been found are convenient for analytical and numerical investigations. It is shown that there is an analogy between the linear Cauchy-Poisson problem for the internal waves under consideration and waves on the surface of a heavy fluid. Solutions of the travelling-wave and standing-wave types are found by expansion in series up to the third order of smallness with respect to the wave amplitude.

The problem of travelling internal waves of finite amplitude /1, 2/ and an analogous problem /3/ taking account of the difference in the velocity of the upper and lower flows have previously been considered. A solution has been given /4/ of the problem of the free finite oscillations of the interface between two unbounded heavy fluids which are at rest at infinity.

1. Formulation of the problem and method of solution. We consider the plane-parallel potential wave flows of a heavy ideal incompressible fluid in two semi-infinite domains Ω_+ and Ω_- which are divided by a periodic curve

$$X = x(s,t),\ Y = y(s,t) \tag{1.1}$$
$$x(s+s_0,t) = \lambda + x(s,t),\ y(s+s_0,t) = y(s,t)$$

where s is a parameter which determines the length of an arc of the curve, s_0 is the length of the arc for a single period of the wave and $\lambda = 2\pi/k$ is the wavelength. The X-axis is directed along the mean level of the boundary of separation, the Y-axis is directed vertically upwards and c_+, c_-, ρ_+ and ρ_- are the unperturbed velocities and densities of the upper flow (the plus index) and the lower flow (the minus index).

It is assumed that the mean level of the interface does not change, that is,

$$\int_0^\lambda y\,dx = 0 \tag{1.2}$$

The velocity fields $\mathbf{v}_+$ and $\mathbf{v}_-$ in the domains Ω_+ and Ω_- respectively can be represented in the form

$$\mathbf{v}_+ = \mathbf{c}_+ + \nabla\varphi_+,\ \mathbf{v}_- = \mathbf{c}_- + \nabla\varphi_- \tag{1.3}$$

where the velocities $\nabla\varphi_+$ and $\nabla\varphi_-$ tend to zero when $Y\to+\infty$ and $Y\to-\infty$ respectively and the potentials φ_+ and φ_- have a period λ with respect to the variable X. It is seen that the mean value of the potentials along a horizontal interval $Y = Y_0$ $(0\leqslant X\leqslant\lambda)$ is independent of Y

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$$\bar{\varphi}_\pm = \frac{1}{\lambda}\int\limits_0^\lambda \varphi_\pm(X,Y)\,dX, \quad \frac{d\bar{\varphi}_\pm}{dY} = 0 \tag{1.4}$$

In fact, by differentiating (1.4) with respect to Y, we obtain the flow rate of the fluid across a horizontal segment, which is equal to zero by virtue of (1.2). Hence, the mean constant quantities $\bar{\varphi}_+$ and $\bar{\varphi}_-$ may be considered to be equal to the limiting values of the corresponding potentials when $Y \to +\infty$ and $Y \to -\infty$.

Let us project the velocities (1.3) onto the normal $\mathbf{n}$, the components of which are

$$n_x = -\partial y/\partial s, \; n_y = \partial x/\partial s \tag{1.5}$$

and write the conditions for the continuity of the normal velocity v on the boundary

$$v = -c_+\partial y/\partial s + \partial\varphi_+/\partial n = -c_-\partial y/\partial s + \partial\varphi_-/\partial n \tag{1.6}$$

The potentials φ_+ and φ_- satisfy Laplace's equation in the domains Ω_+ and Ω_- respectively and their normal velocities and values on the boundary are therefore connected by the linear relationships

$$\mathbf{A}\partial\varphi_\pm/\partial n - \mathbf{B}\varphi_\pm \mp {}^1/_2\,(\varphi_\pm - \bar{\varphi}_\pm) \tag{1.7}$$

where **A** and **B** are linear operators whose form and properties will be indicated in Sect.3. Relationships (1.6) and (1.7) are four equations for determining the five functions $v, \partial\varphi_+/\partial n, \partial\varphi_-/\partial n, \varphi_+$ and φ_- which are defined on a known surface. The missing fifth equation follows from the condition for the continuity of the pressure.

The idea of applying integral relatinships of the type (1.7) has previously been used in many papers /5/ and, in particular, for the numerical investigation of waves on the surface of a heavy fluid. Below, this method is extended to the study of internal waves. The direct implementation of the method is difficult since it requires the solution of a complex system of integral equations. In Sect.2, this system is solved analytically for three of the unknown functions and a system of two equations in the two remaining functions will therefore be obtained.

2. ***Solution of the system of integral equations.*** Instead of the functions φ_+ and φ_- and the numbers c_+ and c_-, we introduce φ, f, c and Δc using the relationships

$$\varphi_\mp = \varphi \pm {}^1/_2 f, \; c_\mp = c \pm {}^1/_2\Delta c \tag{2.1}$$

We substitute expressions (1.6) and (2.1) into system (1.7)

$$\mathbf{A}\,[v + (c \pm {}^1/_2\Delta c)\,\partial y/\partial s] - \mathbf{B}\,[\varphi \pm {}^1/_2 f] = \mp{}^1/_2\,(\varphi \pm {}^1/_2 f - \bar{\varphi}_\mp) \tag{2.2}$$

and solve system (2.2) for the functions v and φ.

In order to do this, we introduce the flow functions ψ_- and ψ_+, which correspond to φ_- and φ_+, and write the Cauchy-Riemann conditions on the boundary for the analytical functions $\varphi_\pm + i\psi_\pm$

$$\partial\psi_\pm/\partial s = -\partial\varphi_\pm/\partial n, \; \partial\psi_\pm/\partial n = \partial\varphi_\pm/\partial s \tag{2.3}$$

Relationships (1.7) also hold in the case of $\psi_\pm$ and, using (2.3), these can be represented in the following form:

$$\mathbf{A}\,(\partial\varphi/\partial s \mp {}^1/_2\partial f/\partial s) - \mathbf{B}\psi_\pm = {}^1/_2\,(\psi_\pm - \bar{\psi}_\pm) \tag{2.4}$$

It follows from (2.3) and (1.6) that

$$\partial\,(\psi_- - \psi_+)/\partial s = -\partial\,(\varphi_- - \varphi_+)/\partial n = \Delta c\,\partial y/\partial s \tag{2.5}$$
$$\psi_- - \psi_+ = \Delta cy + \bar{\psi}_- - \bar{\psi}_+, \; \psi_\pm = \psi \pm {}^1/_2\,(-\Delta cy + \bar{\psi}_- - \bar{\psi}_+)$$

By substituting the latter relationship into (2.4), we obtain a system which is analogous to (2.2)

$$\mathbf{A}\,(\partial\varphi/\partial s \mp {}^1/_2\partial f/\partial s) - \mathbf{B}\,(\psi \pm {}^1/_2\Delta cy) = \mp{}^1/_2\,(\psi \mp {}^1/_2\Delta cy - \bar{\psi}) \tag{2.6}$$

From the systems of Eqs.(2.2) and (2.6), we find

$$\varphi = \bar{\varphi} + \mathbf{B}f - \Delta c\mathbf{A}\partial y/\partial s, \; \psi = \bar{\psi} - \mathbf{A}\partial f/\partial s - \Delta c\mathbf{B}y \tag{2.7}$$

From (1.6) and (2.3), v can be expressed in terms of ψ

$$v + c\frac{\partial y}{\partial s} = \frac{1}{2}\frac{\partial}{\partial n}(\varphi_- + \varphi_+) = -\frac{\partial\psi}{\partial s}$$

whereupon we find that

$$v + c\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}\left(\mathbf{A}\frac{\partial f}{\partial s} + \Delta c \mathbf{B} y\right) \tag{2.8}$$

When its solutions, which are defined by formulae (2.7) and (2.8), are substituted into the system of Eqs.(2.2) and (2.6), we obviously obtain the same equalities. The following identities, which reveal the link between the $\mathbf{A}$ and $\mathbf{B}$ and the properties, follow from this:

$$\mathbf{A}\frac{\partial}{\partial s}\mathbf{A}\frac{\partial f}{\partial s} - \mathbf{B}^2 f + \frac{1}{4}(f - \bar{f}) = 0 \tag{2.9}$$
$$\mathbf{A}\frac{\partial}{\partial s}\mathbf{B} f + \mathbf{B}\mathbf{A}\frac{\partial f}{\partial s} = 0$$

The identities (2.9) also hold for any functions $y(s)$ and $f(s)$.

3. ***The operators A and B.*** Let us take the period of the wave motion being considered as being equal to 2π. The kernels of the operators $\mathbf{A}$ and $\mathbf{B}$ are expressed in terms of Green's function W for a semi-infinite strip of width 2π bounded by the curve $x(s, t)$, $y(s, t)$ with a period in x equal to 2π:

$$W = {}^1\!/_2 \ln\left[2(\operatorname{ch}\bar{y} - \cos\bar{x})\right] \tag{3.1}$$
$$\frac{\partial W}{\partial n'} = -\frac{\partial y(s')}{\partial s'}\frac{\partial W}{\partial \bar{x}} + \frac{\partial x(s')}{\partial s'}\frac{\partial W}{\partial \bar{y}}$$
$$\bar{x} = x(s') - x(s), \quad \bar{y} = y(s') - y(s)$$
$$\mathbf{A} f = \frac{1}{2\pi}\int_0^{s_0} W f(s')\,ds', \quad \mathbf{B} f = \frac{1}{2\pi}\int_0^{s_0}\frac{\partial W}{\partial n'}(f(s') - f(s))\,ds'$$

By expanding Green's function in powers of a small parameter

$$W = \ln\left|2\sin\frac{\bar{x}}{2}\right| + \frac{1}{8}\bar{y}^2/\sin^2\frac{\bar{x}}{2} + \ldots$$
$$\ln\left|2\sin\frac{\bar{x}}{2}\right| = -\cos\bar{x} - \frac{1}{2}\cos 2\bar{x} - \ldots - \frac{1}{n}\cos n\bar{x} - \ldots$$

it is possible to find the expansion of the operators in powers of $\bar{y}$ in the form

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_2 + \ldots, \quad \mathbf{B} = \mathbf{B}_1 + \ldots \tag{3.2}$$

where the index is equal to the order of smallness with respect to $\bar{y}$ of the corresponding integral operator. On substituting expansion (3.2) into (2.9), we get the identities for the operators $\mathbf{A}_0, \mathbf{B}_1, \ldots$

$$\mathbf{A}_0\frac{\partial}{\partial x}\mathbf{A}_0\frac{\partial f}{\partial x} + \frac{1}{4}(f - \bar{f}) = 0, \quad \mathbf{A}_0\frac{\partial}{\partial x}\mathbf{B}_1 f + \mathbf{B}_1\mathbf{A}_0\frac{\partial f}{\partial x} = 0 \tag{3.3}$$

The operator $\mathbf{A}_0$ is associated with the Hilbert operator $\mathbf{H}$ which is well-known in the theory of integral operators /6/

$$\mathbf{H} f = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\bar{x}}{2} f(x')\,dx' = -2\mathbf{A}_0\frac{\partial f}{\partial x} \tag{3.4}$$

It should be noted that all of the formulae obtained in Sect.3, extend to the case of an aperiodic wave motion with a boundary $y(x, t)$ and a function $f(x, t)$ defined in an infinite interval $x \in (-\infty, \infty)$ which satisfy certain smoothness conditions. Here, W will be a Green's function for the semi-infinite domain: $W = {}^1\!/_2 \ln(\bar{x}^2 + \bar{y}^2)$. The operators $\mathbf{A}_0$ and $\mathbf{H}$ are:

$$\mathbf{A}_0 f = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln|x' - x|\,f(x')\,dx', \quad \mathbf{H} f = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x')(x' - x)^{-1}\,dx' \tag{3.5}$$
$$\mathbf{H}^2 f = -f, \quad \mathbf{H} f = -2\mathbf{A}_0 f$$

4. ***Dynamical condition.*** Let us write the Cauchy-Lagrange integral for the lower layer of ideal fluid of density ρ_- at a point on the interface

$$p_- + \rho_-\left[\frac{\partial'\varphi_-}{\partial t} + \frac{1}{2}\left(\frac{\partial\varphi_-}{\partial s}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi_-}{\partial n}\right)^2 + gy\right] = b_-(t) \tag{4.1}$$

Let us now pass to the absolute coordinate system in which the flow has a velocity c_-

$$\frac{\partial\varphi_-}{\partial n} = v + c_-\frac{\partial y}{\partial s}, \quad \frac{\partial'\varphi_-}{\partial t} = \frac{\partial\varphi_-}{\partial t} + c_-\frac{\partial\varphi_-}{\partial x} \tag{4.2}$$

We now express the derivative $\partial/\partial t$ at constant Eulerian coordinates in terms of the partial derivative $\partial^l/\partial t$ at a point which moves along the boundary with a tangential velocity u and the derivative with respect to x in terms of the derivatives with respect to s and n

$$\frac{\partial\varphi_-}{\partial t} = \frac{\partial^l\varphi_-}{\partial t} - u\frac{\partial\varphi_-}{\partial s} - v\frac{\partial\varphi_-}{\partial n}, \quad \frac{\partial\varphi_-}{\partial x} = \frac{\partial\varphi_-}{\partial s}\frac{\partial x}{\partial s} - \frac{\partial\varphi_-}{\partial n}\frac{\partial y}{\partial s}$$

Whereupon, using (4.2), we get

$$\frac{\partial'\varphi_-}{\partial t} = \frac{\partial^l\varphi_-}{\partial t} - u\frac{\partial\varphi_-}{\partial s} + c_-\frac{\partial\varphi_-}{\partial s}\frac{\partial x}{\partial s} - \left(\frac{\partial\varphi_-}{\partial n}\right)^2 \tag{4.3}$$

Substituting (4.3) into (4.1) and using (4.2), we obtain an expression for the pressure p_- in the absolute system of coordinates in terms of a function of the arguments s and t which define the position of the point on the boundary of separation of the flows. A similar expression also holds for p_+

$$p_\pm(s,t) = b_\pm - \rho_\pm\left[\frac{\partial^l\varphi_\pm}{\partial t} - u\frac{\partial\varphi_\pm}{\partial s} + \frac{1}{2}\left(\frac{\partial\varphi_\pm}{\partial s}\right)^2 - \frac{1}{2}v^2 + gy + c_\pm\left(\frac{\partial x}{\partial s}\frac{\partial\varphi_\pm}{\partial s} - v\frac{\partial y}{\partial s}\right) - \frac{1}{2}c_\pm^2\left(\frac{\partial y}{\partial s}\right)^2\right] \tag{4.4}$$

The pressure continuity condition $p_+ - p_- = 0$ yields the following closing equation. Subject to the condition that the inhomogeneity in the density is small

$$\Delta\rho/\rho \ll 1, \quad \rho = {}^1/_2(\rho_- + \rho_+), \quad \Delta\rho = \rho_- - \rho_+ \tag{4.5}$$

the pressure continuity equation on the interface takes the form

$$\frac{\partial^l f}{\partial t} = -\frac{\Delta\rho}{\rho}gy + \left(u - c\frac{\partial x}{\partial s}\right)\frac{\partial f}{\partial s} + \Delta c\left(v + c\frac{\partial y}{\partial s}\right)\frac{\partial y}{\partial s} - \frac{\partial\varphi}{\partial s}\left(\frac{\partial f}{\partial s} + \Delta c\frac{\partial x}{\partial s}\right) + \Delta b \tag{4.6}$$

By adding the equations for the x, y coordinates of the interface

$$\frac{\partial^l x}{\partial t} = -v\frac{\partial y}{\partial s} + u\frac{\partial x}{\partial s}, \quad \frac{\partial^l y}{\partial t} = v\frac{\partial x}{\partial s} + u\frac{\partial y}{\partial s} \tag{4.7}$$

and, also, expressions (2.7) and (2.8) for φ and v

$$\varphi = \bar{\varphi} + \mathbf{B}f - \Delta c\mathbf{A}\frac{\partial y}{\partial s}, \quad v = \frac{\partial}{\partial s}\left(\mathbf{A}|\frac{\partial f}{\partial s} + \Delta c\mathbf{B}y - cy\right) \tag{4.8}$$

we obtain the complete system of equations for determining the functions $f(l,t)$, $x(l,t)$ and $y(l,t)$, where $l(s,t)$ is an arbitrary parameter which determines the position of a point on the boundary of separation. The tangential velocity $u(l,t)$ depends on the choice of the parameter l, that is, on the distribution of the points x, y on the boundary. If one starts out from considerations of the stability of the numerical scheme, it is advisable /7/ to choose $l = s$, and the points x, y will then be uniformly distributed along the length.

We mean by $\partial^l/\partial t$ the partial derivative of functions with respect to time for constant l.

For the analytical investigation of the problem it is convenient to choose $l = x$; then, from (4.6) and (4.7) we get a system of equations in the two functions $f(x,t)$ and $y(x,t)$

$$\frac{\partial f}{\partial t} = \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{-1}\left[\left(\frac{\partial f}{\partial x} + \Delta c\right)\left(v_y\frac{\partial y}{\partial x} - \frac{\partial\varphi}{\partial x}\right) - \right. \tag{4.9}$$

$$c\frac{\partial f}{\partial s} + \Delta cc\left(\frac{\partial y}{\partial s}\right)^2\Big] - \frac{\Delta\rho}{\rho} gy + \Delta b$$

$$\frac{\partial y}{\partial t} = v_y$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}\left(\mathbf{B}f - \Delta c\mathbf{A}\frac{\partial y}{\partial s}\right)$$

$$v_y = \frac{\partial}{\partial x}\left(\mathbf{A}\frac{\partial f}{\partial s} + \Delta c\mathbf{B}y - cy\right)$$

The system of Eqs.(4.9), or system (4.6), (4.7) which is equivalent to it, describes the Cauchy-Poisson problem for internal waves of finite amplitude under the single constraint that $\Delta\rho/\rho \ll 1$.

5. ***The linear Cauchy-Poisson problem.*** By separating the terms which are linear in f and y in the system of Eqs.(4.9), we get

$$\frac{\partial^c f}{\partial t} = -\frac{\Delta\rho}{\rho} gy - \frac{1}{2}\Delta c^2 \mathbf{H}\frac{\partial y}{\partial x} \tag{5.1}$$

$$\frac{\partial^c y}{\partial t} = -\frac{1}{2}\mathbf{H}\frac{\partial f}{\partial x}, \quad \left(\frac{\partial^c}{\partial t} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$$

where $\mathbf{H}$ is a Hilbert operator which is defined according to (3.4) and (3.5). In order to obtain the solution of system (5.1), a knowledge of the initial conditions for the functions f and y is required.

Let us compare the Cauchy-Poisson problem for internal waves, which has been formulated, with the analogous problem for surface waves. It is well-known /8/ that, to solve the latter problem, it is necessary to find the potential Φ which is harmonic in the lower half plane $Y \leqslant 0$ and the function $\eta(X, t)$ which, when $Y = 0$, satisfy the conditions

$$\frac{\partial'\Phi}{\partial t'} = -g\eta, \quad \frac{\partial'\eta}{\partial t'} = \frac{\partial\Phi}{\partial Y} = -\mathbf{H}\frac{\partial\Phi}{\partial X}\left(\frac{\partial'}{\partial t'} = \frac{\partial}{\partial t'} + c'\frac{\partial}{\partial X}\right) \tag{5.2}$$

The latter equality can be obtained from the integral relationship (2.4) on the boundary of the half plane.

If the velocities of the flows are equal ($\Delta c = 0$), problem (5.1) is equivalent to problem (5.2) and, with the substitution

$$\Phi = f/\mu,\ t' = t\mu,\ c' = c/\mu,\ \eta = y,\ \mu = \sqrt{2\Delta\rho/\rho} \tag{5.3}$$

the solution of the Cauchy-Poisson problem for surface waves yields the solution of the problem for internal waves.

In the general case when $\Delta c \neq 0$, y can be eliminated from sytem (5.1) and we then get an equation for the function $f(x, t)$

$$\frac{\partial^{c^2} f}{\partial t^2} + \frac{1}{4}\Delta c^2\frac{\partial^2 f}{\partial x^2} - \frac{1}{2}\frac{\Delta\rho}{\rho} g\mathbf{H}\frac{\partial f}{\partial x} = 0 \tag{5.4}$$

In a similar manner, it is possible to obtain the same equation for $y(x, t)$. If the relationships

$$\mathbf{H}(\cos kx) = -\sin kx,\ \mathbf{H}(\sin kx) = \cos kx \tag{5.5}$$

are made use of, it is possible to obtain the solution of Eq.(5.4) in the form of a standing wave and a progressive wave. In the coordinate system in which $c = 0$, these solutions have the form

$$f = \sin\sigma t\cos kx,\ f = \sin(kx - \sigma t) \tag{5.6}$$

$$\sigma = \pm\sqrt{\frac{1}{2}\frac{\Delta\rho}{\rho} gk - \frac{1}{4}k^2\Delta c^2}$$

The exact frequency of the linear standing waves is defined by the formula /9/

$$\sigma = \pm\sqrt{\frac{1}{2}\frac{\Delta\rho}{\rho} gk - \frac{1}{4}k^2\Delta c^2\left(1 - \frac{1}{4}\left(\frac{\Delta\rho}{\rho}\right)^2\right)} \tag{5.7}$$

which has a difference of the second order of smallness with respect to the parameter $\Delta\rho/\rho$.

6. Progressive and standing waves. In order to determine the progressive waves in Eqs. (4.9), it follows that one should put $v_y = \partial y/\partial t = 0$ and $\partial f/\partial t = 0$, whereupon the system of equations then takes the form

$$cy = \mathbf{A}\frac{\partial}{\partial s} + \Delta c\mathbf{B}y \tag{6.1}$$

$$\left(\frac{\partial f}{\partial x} + \Delta c\right)\frac{\partial}{\partial x}\left(\Delta c\mathbf{A}\frac{\partial y}{\partial s} - \mathbf{B}f\right) - c\frac{\partial f}{\partial x} + c\Delta c\left(\frac{\partial y}{\partial x}\right)^2 +$$
$$\left(\Delta b - \frac{\Delta\rho}{\rho}gy\right)\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right) = 0$$

The solution of system (6.1), which has been generated by the linear approximation (5.6), apart from terms of the third order, has the form

$$ky = \varepsilon\cos kx + \varepsilon^2 y_2\cos 2kx + \varepsilon^3 y_3(-\cos kx + \cos 3kx) \tag{6.2}$$
$$c^2 = c_0^2 + \varepsilon^2 c_2 \; (c_0^2 = q - {}^1/_4\Delta c^2, \;\; c_2 = {}^1/_2 q + {}^1/_2 q^{-1}c_0^2\Delta c^2)$$
$$y_2 = {}^1/_2 q^{-1}c_0\Delta c, \quad y_3 = {}^1/_2 c_0^2\Delta c^2 q^{-2} - {}^1/_3$$
$$q = {}^1/_2 gk^{-1}\Delta\rho/\rho$$

An analytical solution of the progressive-wave type up to the second order of smallness with respect to the wave amplitude ε was found in /3/, allowing for a shift in the velocity of one flow with respect to the other (that is, $\Delta c \neq 0$ in our formulation). Apart from the choice of the coordinate system, the solution (6.2) agrees with the solution given in /3/.

In order to determine the standing waves we put $c = 0$ in (4.9) and the system of equations takes the form

$$\frac{\partial f}{\partial t} = \left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{-1}\left(\frac{\partial f}{\partial x} + \Delta c\right)\left(v_y\frac{\partial y}{\partial x} - \frac{\partial\varphi}{\partial x}\right) - \frac{\Delta\rho}{\rho}gy + \Delta b, \qquad \frac{\partial y}{\partial t} = v_y \tag{6.3}$$
$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\left(\mathbf{B}f - \Delta c\mathbf{A}\frac{\partial y}{\partial s}\right)$$
$$v_y = \frac{\partial}{\partial x}\left(\mathbf{A}\frac{\partial f}{\partial s} + \Delta c\mathbf{B}y\right)$$

The solution of system (6.3) for the wave profile has the form

$$ky = \varepsilon\cos\sigma t\cos kx + \varepsilon^2 y_{22}\sin 2\sigma t\sin 2kx + \tag{6.4}$$
$$\varepsilon^3\left[(y_{11}\cos\sigma t + y_{31}\cos 3\sigma t)\cos kx + (y_{13}\cos\sigma t + y_{33}\cos 3\sigma t)\times\right.$$
$$\left.\cos 3kx\right]$$
$$\sigma^2 = \sigma_0^2 + \sigma_2\varepsilon^2$$
$$\sigma_0^2 = \left(q - \frac{1}{4}\Delta c^2\right)k^2, \quad \sigma_2 = -\frac{1}{16}(q - 4\Delta c^2 + q^{-1}\Delta c^4)k^2$$
$$y_{22} = {}^1/_2 q^{-1}k^{-1}\Delta c\sigma_0$$
$$y_{11} = \frac{9}{64} - \frac{5}{16}q^{-1}\Delta c^2 + \frac{1}{8}q^{-2}\Delta c^4 - \frac{1}{256}\Delta c^2 k^2\sigma_0^{-2}$$
$$y_{31} = -\frac{1}{64} + \frac{1}{16}q^{-1}\Delta c^2 + \frac{1}{256}\Delta c^2 k^2\sigma_0^{-2}$$
$$y_{13} = -\frac{3}{32}, \quad y_{33} = -\frac{1}{32} + \frac{1}{2}q^{-2}k^{-2}\Delta c^2\sigma_0^2$$

In the special case when $\Delta c = 0$, we have

$$ky = \varepsilon\cos\sigma t\cos kx + \varepsilon^3\left[\left(\frac{9}{64}\cos\sigma t - \frac{1}{64}\cos 3\sigma t\right)\cos kx + \right. \tag{6.5}$$
$$\left.\left(-\frac{3}{32}\cos\sigma t - \frac{1}{32}\cos 3\sigma t\right)\cos 3kx\right]$$
$$\sigma^2 = \frac{1}{2}\frac{\Delta\rho}{\rho}gk - \frac{1}{32}\frac{\Delta\rho}{\rho}gk\varepsilon^2$$

We note that, in expression (6.5) for the wave profile $y(x, t)$, there is no term in ε^2 and, moreover, it can be seen from Eqs.(4.9) that, when $\Delta c = 0$, the wave is symmetrical about the abscissa in the main approximation $\Delta\rho/\rho \ll 1$. The particular solution of (6.5) when $\Delta c = 0$ agrees with the solution in /4/ where an expansion up to the third order of smallness in the parameter $\zeta = \varepsilon + {}^7/_{64}\varepsilon^3$ was carried out. The expansion in series up to the fifth order in ε in /10/ is also found to be in accord with (6.5).

REFERENCES

1. KOCHIN N.E., Exact determination of the steady-state waves of finite amplitude on the surface of separation of two fluids of finite depth, Sobr. Soch., 2, Izd. Akad. Nauk SSSR, Moscow-Leningrad, 1949.
2. SINHA S.R.P., The exact theory of steady-state waves on a free surface and the surface of separation of two fluids, Dokl. Akad. Nauk SSSR, 168, 1, 1966.
3. SAFFMAN P.G. and YUEN H.C., Finite-amplitude interfacial waves in the presence of a current. J. Fluid Mech. 123, 1982.
4. SEKERZH-ZEN'KOVICH YA.I., On the theory of the finite oscillations of the surface of separation of two unbounded heavy fluids with different densities, Dokl. Akad. Nauk SSSR, 36, 1, 1961.
5. BREBBIA C., TELLES G. and WROBEL L., Boundary Element Methods, Mir, Moscow, 1987.
6. BATEMAN H. and ERDELYI A., Tables of Integral Transformations. 2, Bessel Transformations. Integrals of Special Functions, Nauka, Moscow, 1970.
7. VOINOV V.V., VOINOV O.V. and PETROV A.G., A method of calculating the potential flow round a solid of rotation by an incompressible fluid, Zh. vychisl. Mat. mat. Fiz., 14, 3, 1974.
8. STRETENSKII L.N., Theory of the Wave Motions of a Fluid, Nauka, Moscow, 1977.
9. LAMB G., Hydrodynamics, Gostekhizd. Moscow-Leningrad, 1947.
10. ROTTMAN J.W., Steep standing waves at a fluid interface. J. Fluid Mech. 124, 1982.

Translated by E.L.S.

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LINEAR WAVES IN A FLUID FLOW WITH CONSTANT VORTICITY LOCATED UNDER AN ICE BLANKET*

A.V. MARCHENKO and I.V. PROKHOROV

The linear dynamics of periodic waves on the surface of a fluid layer of finite depth located under an ice blanket which is simulated by an elastic plate is considered. The fluid particles in the unperturbed state move at a constant horizontal velocity, the profile of which has a linear shift along the vertical. It is shown that several type of waves exist which propagate at the same frequency. The number of waves depends on the frequency, the flow parameters in the fluid and the physico-mechanical parameters of the ice blanket. The problem of the diffraction of waves of fixed frequency on the edge of a semi-infinite elastic plate which floats on the surface of the fluid is considered. The problem is reduced to the solution of Laplace's equation in the strip with specified asymptotic forms at infinity and with boundary conditions on the sides of the strip which have a discontinuity at a point corresponding to the edge of the ice and contact-boundary conditions on the edge of the plate. The solution is constructed using the Wiener-Hopf method. The reflection and transmission coefficients of the waves across the edge of the plate are determined. The results obtained are analysed using the actual parameters of sea ice.

In investigations of the dynamics of waves in a fluid layer with a constant vorticity

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